The Correlated Binomial Distribution - Part II

Gary Schurman MBE, CFA

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In Part I we developed the mathematics for the correlated binomial distribution. To demonstrate the mathematics we solved a simple bond portfolio problem where the portfolio consisted of three bonds. With three bonds it was easy to calculate the probabilities of zero defaults, one default, two defaults and three defaults. As we increase the number of bonds in our portfolio the math becomes more and more tedious as the requisite number of calculations increase. In this part we will develop several closed-form equations that will come in handy when evaluating portfolios of sufficient size. As is usually the case we will start with a hypothetical problem...

Our Hypothetical Problem

Imagine that we have a bond portfolio that consists of twenty bonds each with a principal balance of $1,000,000 and each with a maturity date one year hence. If at maturity the bond does not default we will receive the principal balance plus 15.00% simple interest. If at maturity the bond does default we will receive the recovery on that bond which is expected to be 40% of the principal balance. Each bond has an annual unconditional default probability of 0.08 and a pairwise default correlation of 0.10.

Question 1: What is the default probability distribution for our bond portfolio?

Question 2: What is our expected total return?

Question 3: What is the probability of realizing seven or more defaults?

Closed-Form Equation For Conditional Probability

In Part I we defined the variable p to be the unconditional probability of default. We also defined the conditional probabilities of default given one, two and three prior defaults to be...

$$p_0 = p \tag{1}$$

$$p_1 = p_0\,(1-\theta_{1,2}) + \theta_{1,2} \tag{2}$$

$$p_2 = p_1\,(1-\theta_{2,3}) + \theta_{2,3} \tag{3}$$

$$p_3 = p_2\,(1-\theta_{3,4}) + \theta_{3,4} \tag{4}$$

Note that in the conditional probability equations above in order to calculate the conditional probability of default given n prior defaults we must first calculate the $n-1$ conditional probabilities of default given $\{1,2,3,...,n-1\}$ prior defaults. We will develop a closed-form equation for conditional default that is not iterative as are Equations (2), (3) and (4) above.

The first thing that we will do is drop the subscripts from the conditional correlation variable theta and assume that conditional correlation does not change as defaults increase. After making this change we take Equation (4) and replace p_2, which is the conditional probability of default given two prior defaults, with Equation (3). The new equation for the conditional probability of default given three prior defaults is...

$$\begin{aligned} p_3 &= (p_1(1-\theta)+\theta)\,(1-\theta)+\theta \\ &= p_1(1-\theta)^2 + \theta\,(1-\theta)+\theta \end{aligned} \tag{5}$$

We now replace p_1, which is the conditional probability of default given one prior default, in Equation (5) with Equation (2). The revised equation for the conditional probability of default given three prior defaults is...

$$\begin{aligned}
p_3 &= (p_0(1-\theta)+\theta)\,(1-\theta)^2+\theta\,(1-\theta)+\theta \\
&= p_0(1-\theta)^3+\theta\,(1-\theta)^2+\theta\,(1-\theta)+\theta \\
&= p_0(1-\theta)^3+\theta\,(1-\theta)^2+\theta\,(1-\theta)^1+\theta\,(1-\theta)^0 \\
&= p_0(1-\theta)^3+\theta\,\{(1-\theta)^2+(1-\theta)^1+(1-\theta)^0\}
\end{aligned} \tag{6}$$

We now replace p_0, which is the unconditional probability of default, in Equation (6) with Equation (1). The revised equation for the conditional probability of default given three prior defaults is...

$$p_3 = p\,(1-\theta)^3+\theta\,\{(1-\theta)^2+(1-\theta)^1+(1-\theta)^0\} \tag{7}$$

Note that if one minus theta is less than one and greater than zero then the following geometric series holds...

$$\sum_{i=0}^{\infty}(1-\theta)^i = \frac{1}{1-(1-\theta)} = \frac{1}{\theta} \tag{8}$$

Also note that if one minus theta is less than one and greater than zero, and the variable j is greater than zero then the following geometric series holds...

$$\sum_{i=j}^{\infty}(1-\theta)^i = (1-\theta)^j\sum_{i=0}^{\infty}(1-\theta)^i = \frac{(1-\theta)^j}{\theta} \tag{9}$$

If we combine Equations (8) and (9) then we can make the following definition...

$$\sum_{i=0}^{j-1}(1-\theta)^i = \sum_{i=0}^{\infty}(1-\theta)^i - \sum_{i=j}^{\infty}(1-\theta)^i = \frac{1}{\theta} - \frac{(1-\theta)^j}{\theta} = \frac{1-(1-\theta)^j}{\theta} \tag{10}$$

We will define default correlation to be a number greater than zero and less than one. Using Equation (10) we can now rewrite Equation (7) above as...

$$\begin{aligned}
p_3 &= p\,(1-\theta)^3+\theta\sum_{i=0}^{2}(1-\theta)^i \\
&= p\,(1-\theta)^3+\theta\left\{\sum_{i=0}^{\infty}(1-\theta)^i-\sum_{i=3}^{\infty}(1-\theta)^i\right\} \\
&= p\,(1-\theta)^3+\theta\left\{\frac{1-(1-\theta)^j}{\theta}\right\} \\
&= p\,(1-\theta)^3+(1-(1-\theta)^3)
\end{aligned} \tag{11}$$

Using Equation (11) as a template we can write the equation for conditional probability given k prior defaults in closed-form as...

$$p_k = p\,(1-\theta)^k+(1-(1-\theta)^k) \tag{12}$$

Closed-Form Equation For Zero Defaults

In Part I we determined that the equation for the probability that none of the three bonds would default (i.e. $X_1=0$, $X_2=0$ and $X_3=0$) was...

$$\begin{aligned}
P[0] &= \mathbb{E}[(1-X_1)(1-X_2)(1-X_3)] \\
&= 1-3\,p_0+3\,p_0\,p_1-p_0\,p_1\,p_2
\end{aligned} \tag{13}$$

Equation (13) above has only three bonds so the solution was easy to obtain. Note that the hypothetical problem above has twenty bonds rather than three. The equation for the conditional probability of zero defaults in a portfolio of twenty bonds is...

$$P[0] = \mathbb{E}[(1-X_1)(1-X_2)(1-X_3)(1-X_4)(1-X_5)(1-X_6)...(1-X_{19})(1-X_{20})] \tag{14}$$

To multiply out and solve Equation (14) above would definitely be a chore that we would like to avoid. What we need is closed-form equation for zero defaults. The closed-form equation for zero defaults out of n bonds happens to be...

$$P[0] = \mathbb{E}\Big[\prod_{j=1}^{n}(1-X_j)\Big] = 1 + \sum_{j=1}^{n}(-1)^j\binom{n}{j}\prod_{i=0}^{j-1}p_i \tag{15}$$

To demonstrate how the equation works we will use Equation (15) to solve Equation (13) above. Per the summation on the right-hand side of the equation above the variable j goes from one to n. Using this equation we get the following results...

$$j = 1 : (-1)^1\binom{3}{1}\prod_{i=0}^{1-1}p_i = -1 \times 3 \times p_0 = -3\,p_0 \tag{16}$$

$$j = 2 : (-1)^2\binom{3}{2}\prod_{i=0}^{2-1}p_i = 1 \times 3 \times p_0\,p_1 = 3\,p_0\,p_1 \tag{17}$$

$$j = 3 : (-1)^3\binom{3}{3}\prod_{i=0}^{3-1}p_i = -1 \times 1 \times p_0\,p_1\,p_2 = -p_0\,p_1\,p_2 \tag{18}$$

If we combine the results of Equations (16), (17) and (18) we get...

$$P[0] = \mathbb{E}\Big[\prod_{j=1}^{3}(1-X_j)\Big] = 1 + \sum_{j=1}^{3}(-1)^j\binom{3}{j}\prod_{i=0}^{j-1}p_i = 1 - 3\,p_0 + 3\,p_0\,p_1 - p_0\,p_1\,p_2 \tag{19}$$

...which is Equation (13) and therefore concludes the proof.

Closed-Form Equation For Multiple Defaults

In Part I we determined that the equation for the probability that one of the three bonds would default (i.e. $[X_1 = 1$, $X_2 = 0$ and $X_3 = 0]$ ...or... $[X_1 = 0$, $X_2 = 1$ and $X_3 = 0]$ ...or... $[X_1 = 0$, $X_2 = 0$ and $X_3 = 1])$ was...

$$\begin{aligned} P[1] &= \mathbb{E}[X_1(1-X_2)(1-X_3)] \\ &= p_0 - 2\,p_0\,p_1 + p_0\,p_1\,p_2 \end{aligned} \tag{20}$$

The equation for the conditional probability of one default in a portfolio of twenty bonds is...

$$P[1] = \mathbb{E}[X_1(1-X_2)(1-X_3)(1-X_4)(1-X_5)(1-X_6)...(1-X_{19})(1-X_{20})] \tag{21}$$

To multiply out and solve Equation (21) above would definitely be a chore that we would like to avoid. In Equation (15) above we presented the closed-form equation for the conditional probability of zero defaults in a portfolio of n bonds. The following equation is the closed-form equation for the conditional probability of k defaults in a portfolio of n bonds...

$$\binom{n}{k}\mathbb{E}\Big[\prod_{j=1}^{k}X_j\prod_{j=k+1}^{n}(1-X_j)\Big] = \binom{n}{k}\sum_{j=0}^{n-k}(-1)^j\binom{n-k}{j}\prod_{i=0}^{j+k-1}p_i \tag{22}$$

To demonstrate how this equation is used we will use Equation (22) to solve Equation (20) above. Per the summation on the right-hand side of the equation above the variable j goes from zero to $n-1$. Using this equation we get the following results...

$$j = 0 : (-1)^0\binom{2}{0}\prod_{i=0}^{0+1-1}p_i = 1 \times 1 \times p_0 = p_0 \tag{23}$$

$$j = 1 : (-1)^1\binom{2}{1}\prod_{i=0}^{1+1-1}p_i = -1 \times 2 \times p_0\,p_1 = -2\,p_0\,p_1 \tag{24}$$

$$j = 2 : (-1)^2\binom{2}{2}\prod_{i=0}^{2+1-1}p_i = 1 \times 1 \times p_0\,p_1\,p_2 = p_0\,p_1\,p_2 \tag{25}$$

If we combine the results of Equations (23), (24) and (25) we get...

$$P[1] = \mathbb{E}\Bigg[\prod_{j=1}^{1} X_j \prod_{j=2}^{3}(1-X_j)\Bigg] = \sum_{j=0}^{3-1}(-1)^j \binom{3-1}{j} \prod_{i=0}^{j+1-1} p_i = p_0 - 2\,p_0\,p_1 + p_0\,p_1\,p_2 \tag{26}$$

...which is Equation (20) and therefore concludes the proof.

The Answer To The Hypothetical Problem

Using Equations (12), (15) and (22) above the answer to Question 1: What is the default probability distribution for our bond portfolio? is...

Default	Survive	Probability	Payoff
0	20	0.4031	23,000,000
1	19	0.2222	22,250,000
2	18	0.1373	21,500,000
3	17	0.0872	20,750,000
4	16	0.0558	20,000,000
5	15	0.0357	19,250,000
6	14	0.0226	18,500,000
7	13	0.0142	17,750,000
8	12	0.0089	17,000,000
9	11	0.0054	16,250,000
10	10	0.0033	15,500,000
11	9	0.0019	14,750,000
12	8	0.0011	14,000,000
13	7	0.0006	13,250,000
14	6	0.0003	12,500,000
15	5	0.0002	11,750,000
16	4	0.0001	11,000,000
17	3	0.0000	10,250,000
18	2	0.0000	9,500,000
19	1	0.0000	8,750,000
20	0	0.0000	8,000,000
Total		1.0000	21,800,000

The answer to Question 2: What is the expected total return? is...

$$\text{Total Return} = \left[\frac{21,800,000 - 20,000,000}{20,000,000} - 1\right] \times 100 = 9.00\% \tag{27}$$

The answer to Question 3: What is the probability of realizing seven or more defaults? is...

$$\text{Prob[D} \geq 7] = 0.0142+0.0089+0.0054+0.0033+0.0019+0.0011+0.0006+0.0003+0.0002+0.0001 = 0.0361 \tag{28}$$